Strengthening Nevanlinna's Five Value Theorem for Difference Operator

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Abstract

In this paper, we generalise the Nevanlinna's five value theorem for difference operator for finite ordered meromorphic functions which in turn may generalise and simplify many earlier results of Nevanlinna theory.

Keywords: Meromorphic functions, Nevanlinna's counting functions, periodic func- tions, c-separated pairs, exceptional pairs

1 Introduction, Definitions and Results

Nevanlinna's five value theorem has always amazed researchers in analysis. It has been generalised in many ways over the years (Yang & Yi, 2003). The recent years have witnessed an increased interest in the study of the Nevanlinna theory for the difference operator. The difference analogue for the classical Nevanlinna theory has first been studied by Halburd and Korhonen in the year 2006. We can see that the difference analogue for non-constant meromorphic functions for exact differences is $f(z+c) - f(z) = \Delta_c f(z) \equiv 0$, for $c \in \mathbb{C}$. So instead of non-constant meromorphic functions f, we will use the meromorphic functions for which $\Delta_c f \not\equiv 0$ throughout the paper. We will further investigate into the topic and try to deduce difference analogues to the classical Nevanlinna Five point theorem. We need to define certain terms first.

Definition 1.1. (Halburd & Korhonen, 2006) Let $a, c \in \mathbb{C}$ and f be a finite-order non-constant meromorphic function. Two points p and q are said to be c-separated a-pairs if $f(p) = f(q) = a$ and $q = p + c$.

In order to establish the difference analogue of the Nevanlinna theory, we need the following definition of an analogous counting function $n_c(r, a)$.

Definition 1.2. (Halburd & Korhonen, 2006) For a non-constant meromorphic function f, $c, a \in \mathbb{C}$, $n_c(r, a)$ is the number of points z_0, in $|z| \leq r$, where $f(z_0) = a$ and $f(z_0 + c) = a$, counted according to the number of equal terms in the expansions of $f(z)$ and $f(z+c)$ in a neighbourhood of z_0. Also,

$$N_c(r,a) = \int_0^r \frac{n_c(t,a) - n_c(0,a)}{t}\,dt + n_c(0,a)\log r$$

$$N_c(r,\infty) = \int_0^r \frac{n_c(t,\infty) - n_c(0,\infty)}{t}\,dt + n_c(0,\infty)\log r,$$

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where $n_c(r,)$ is the number of c-separated pole pairs of f , which are exactly the c-separated 0-pairs of $1/f$.

Definition 1.3. (Halburd & Korhonen, 2006) For the integrated counting functions $N(r, a)$ and $N_c(r, a)$, we have

$$\tilde{N}_c(r, a) = N(r, a) - N_c(r, a)$$

which is the number of a-points of f ignoring the c-separated pairs.

The above definition is a generalisation of $\overline{N}(r, f)$ of classical Nevanlinna theory which is the integrated counting function of the number of distinct poles of f in $|z| \leq r$.

Definition 1.4. Let $c \in \mathbb{C}$ and f be a meromorphic function of finite order. Let p be a positive integer and a be a small function of f. Then we define $\tilde{N}_c^{p)}(a, r)$ as the number of a-points of f ignoring at most p number of c-separated pairs of f. Similarly, we define $\tilde{N}_c^{(p}(a, r)$ as the number of a-points of f ignoring at least p number of c-separated pairs of f.

Also, if A is a subset of $\mathbb{C}$, then we define $\tilde{N}_c^{A}(a, r)$ as the number of a-points of f in the set A, ignoring the c-separated pairs.

Definition 1.5. (Hayman, 1964) Let f be a non-constant meromorphic function and let $a \in \mathbb{C} \cup \{\infty\}$. Then,

$$E(a, f) = \{z|\ f(z) = a\}$$

where, each zero is counted with its multiplicity and

$$\overline{E}(a, f) = \{z|\ f(z) = a\},$$

$$\overline{E}_k = \{z|\ f(z) = a \text{ with multiplicity } \leq k\}$$

where, each zero is counted exactly once.

Analogous to the above definition, we define the following for c-separated pairs.

Definition 1.6. (Halburd & Korhonen, 2006) Let $c \in \mathbb{C}$ and f be a meromorphic function. We define $\tilde{E}_c(a; f)$ to be the set of all a-points of f, ignoring the c-separated pairs.

Having defined the integrated counting function for c-separated pairs, we need to define exceptional paired values of f and write an analogue for the theorem of exceptional values for meromorphic functions.

Definition 1.7. (Halburd & Korhonen, 2006) Let $a \in \mathbb{C}$. Then a is an exceptional paired value of f with separation c if whenever $f(z) = a$, then $f(z + c) = a$ with the same or higher multiplicity for all except at most finitely many a-points of f.

Analogous to the definitions of the index of multiplicity, for a meromorphic function, Halburd and Korhonen defined the c-separated pair index as the following:

Definition 1.8. (Halburd & Korhonen, 2006) If $c \in \mathbb{C}$ and f is a meromorphic function of finite order, then

$$\pi_c(a, f) = \liminf_{r \to \infty} \frac{N_c(r, a)}{T(r, f)},$$

where, a is either a slowly moving periodic function with period c, or $a = \infty$. We also have the following definition

$$\Pi_c(a,f) = 1 - \limsup_{r\to\infty} \frac{\tilde{N}_c(r,a)}{T(r,f)},$$

which is analogous to the definition of

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)}.$$

For standard notations on the classical Nevanlinna theory, we refer the reader to Hayman, Yang & Yi and Yang.

The famous five value theorem of Nevanlinna may be represented as follows:

Theorem 1.9. (Hayman, 1964) Let f and g be two non-constant meromorphic functions and $a_j \in \mathbb{C} \cup \{\infty\}$ be distinct for $j = 1, 2, 3, 4, 5$. If $E_{\infty)}(a;f) = E_{\infty)}(a;g)$ for $j = 1(1)5$ then $f \equiv g$.

In 1976, H. S. Gopalakrishna and S. S. Bhoosnurmath improved Theorem 1.9 in the following manner.

Theorem 1.10. (Gopalakrishna & Bhoosnurmath, 1976) Let f and g be distinct non-constant meromorphic functions. If there exist distinct elements $a_1, a_2, \dots, a_k$ of $\mathbb{C} \cup \{\infty\}$ such that $E_{p_j)}(a;f) = E_{p_j)}(a;g)$ for $j = 1(1)k$ where $p_1, p_2, \dots, p_k$ are positive integers or ∞ with $p_1 \geq p_2 \geq \dots \geq p_k$, then

$$\sum_{j=2}^{k} \frac{p_j}{1+p_j} \leq 2 + \frac{p_1}{1+p_1}.$$

In 2000, Y. Li and J. Qiao improved Theorem 1.9 by considering shared small functions instead of shared values. Their result may be stated as follows:

Theorem 1.11. (Li & Qiao, 2000) Let f and g be non-constant meromorphic functions and a_j be distinct elements in $S(f) \cap S(g)$ for $j = 1(1)5$. If $E_{\infty)}(a_j;f) = E_{\infty)}(a_j;g)$ for $j = 1(1)5$ then $f \equiv g$.

The above theorem was further generalised by Chen Chen and Tsai in the year 2007 in the following

Theorem 1.12. (Chen, Chen & Tsai, 2007) Let f and g be two non-constant meromorphic functions. If $a_1, a_2, \dots, a_k$, $k \geq 5$ are distinct functions in $S(f) \cap S(g)$ such that

$$\overline{E}(a_j,f) \subseteq \overline{E}(a_j,g),$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \overline{N}\left(r,\frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \overline{N}\left(r,\frac{1}{g-a_j}\right)} > \frac{1}{k-3},$$

then $f \equiv g$.

The Nevanlinna's five value theeorem along with the value distribution theory has developed manifolds and in many directions over the years. Researchers have tried to reduce the modes of sharing by defining weighted sharing and other partial sharing methods, taking moving targets or small functions, relaxing the conditions of sharing etc. The necessity of discretization of variable and calculus have further opened up dimensions of the improvement. The difference analogue for the classical Nevanlinna theory was first studied by Halburd and Korhonen in 2006.

We now state the main results of this paper.

Theorem 1.13. Let $c \in \mathbb{C}$ and let f and g be two finite-ordered meromorphic functions. If a_k, $k = 1(1)5$, are five distinct periodic functions of period c in $S(f) \cap S(g)$ such that

$$\tilde{E}_c(a_k; f) \subseteq \tilde{E}_c(a_k; g), \tag{1}$$

and

$$\liminf_{r\to\infty} \frac{\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right)}{\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right)} > \frac{1}{2},$$

then, either $f \equiv g$ or both f and g are periodic with period c.

The next theorem is a generalisation of the previous theorem which is analogous to theorem 1.12.

Theorem 1.14. Let $c \in \mathbb{C}$ and let f and g be two finite-ordered meromorphic functions. If $a_1, a_2, \ldots, a_k$, $k \geq 5$, are distinct periodic functions of period c in $S(f) \cap S(g)$ such that

$$\tilde{E}_c(a_k, f) \subseteq \tilde{E}_c(a_k, g), \tag{2}$$

and

$$\liminf_{r\to\infty} \frac{\sum_{k=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right)}{\sum_{k=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right)} > \frac{1}{k-3},$$

then, either $f \equiv g$ or both f and g are periodic with period c.

The above theorem is a generalisation of theorem 1.13. Putting $k = 5$, we get

$$\liminf_{r\to\infty} \frac{\sum_{k=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right)}{\sum_{k=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right)} > \frac{1}{5-3} = \frac{1}{2}$$

which gives the same condition as that of theorem 1.13.

2 Lemmas

Lemma 2.1. (Halburd & Korhonen, 2006) Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1(z), a_2(z), \ldots, a_q(z)$

be distinct meromorphic periodic functions with period c such that $a_k \in S(f)$ for $k = 1, \ldots, q$. Then

$$(q-1)T(r,f) \leq \tilde{N}_c(r,f) + \sum_{k=1}^{q} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right) + S(r,f)$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

We state a result analogous to the Picard's theorem for classical Nevanlinna theory.

Lemma 2.2. (Halburd & Korhonen, 2006) Let $c \in \mathbb{C}$. If a finite-order meromorphic function f has three exceptional paired values with separation c, then f is a periodic function with period c.

3 Proof of the Main Theorems

Proof of theorem 1.13

Proof. Suppose f is periodic with period c. Then by definition, all the a-points are paired and by the equation (1), g has atleast five exceptional paired values, and hence, it has to be periodic by lemma 2.2.

So, without loss of generality we assume that neither f nor g are periodic with period c and that $f \not\equiv g$. We have, by lemma 2.1 and the properties of $\tilde{N}_c(r,f)$,

$$3T(r,f) \leq \sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right) + S(r,f), \tag{3}$$

outside a set of finite logarithmic measure. Similarly, we have for g,

$$3T(r,g) \leq \sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right) + S(r,g), \tag{4}$$

outside a set of finite logarithmic measure. Since $f \not\equiv g$, by equations (3) and (4), we have,

$$\begin{aligned}
\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right) &\leq \overline{N}\left(r, \frac{1}{f-g}\right) \\
&\leq T(r,f) + T(r,g) + O(1) \\
&\leq \frac{1}{3}\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right) + \frac{1}{3}\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right) + S(r,f) + S(r,g) + O(1) \\
&\leq \frac{1}{3}\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right) + \frac{1}{3}\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right) + o(1),
\end{aligned}$$

from which it follows that

$$\liminf_{r\to\infty} \frac{\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right)}{\sum_{k=1}^{5} \tilde{N}_c\left(r, \frac{1}{g-a_k}\right)} \le \frac{1}{2},$$

which is a contradiction to our assumption. Hence f and g has to be identical. □

Proof of theorem 1.14

Proof. Suppose first that f is periodic with period c. Then all the a-points of f are paired and by the equation (2), g has at least five exceptional paired values. Thus, g has to be periodic by lemma 2.2.

So, assume that neither f nor g are periodic with period c and that $f \not\equiv g$. We have, by lemma 2.1 and the properties of $\tilde{N}_c(r, f)$,

$$(k-2)T(r,f) \le \sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + S(r,f), \qquad (5)$$

outside a set of finite logarithmic measure. Similarly, we have for g,

$$(k-2)T(r,g) \le \sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right) + S(r,g), \qquad (6)$$

outside a set of finite logarithmic measure. Since $f \not\equiv g$, by equations (5) and (6), we have,

$$\begin{aligned}
\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) &\le N\left(r, \frac{1}{f-g}\right) \\
&\le T\left(r, \frac{1}{f-g}\right) \\
&= T(r, f-g) + O(1) \\
&\le T(r, f) + T(r, g) + O(1) \\
&\le \frac{1}{k-2}\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + \frac{1}{k-2}\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right) \\
&\quad + S(r,f) + S(r,g) + O(1) \\
&\le \left(\frac{1}{k-2} + o(1)\right)\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + \left(\frac{1}{k-2} + o(1)\right)\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right),
\end{aligned}$$

from which it follows that

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right)} \le \frac{1}{k-3},$$